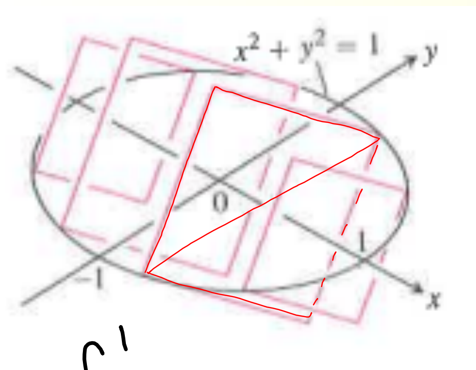
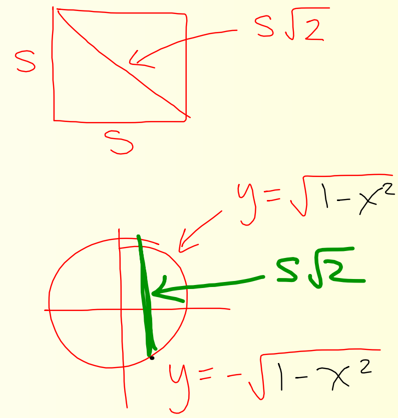
Calculus Warm Up # 6-1

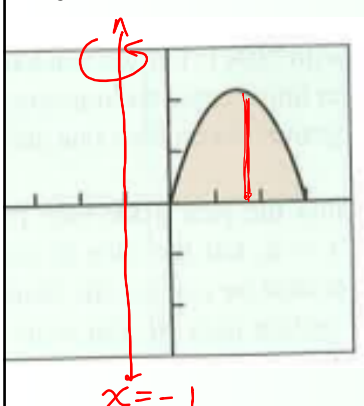
A solid has cross sections perpendicular to the x-axis inside $x^2 + y^2 = 1$. The cross sections are squares with diagonals in the xy plane. Find the volume.

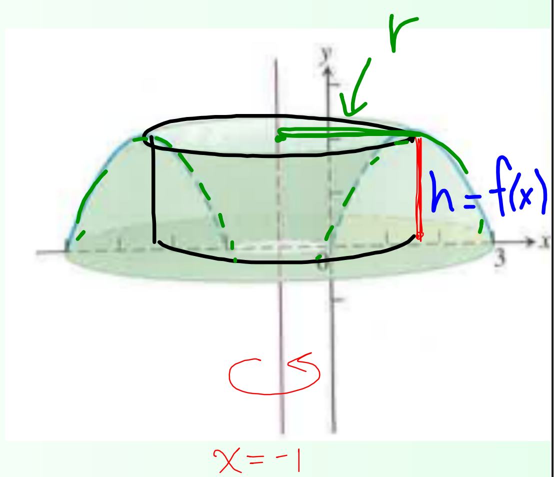




$$\int_{-1}^{1} A(x)\,dx$$

Cylindrical Shells:





$$\int A(x)\,dx$$

rectangles

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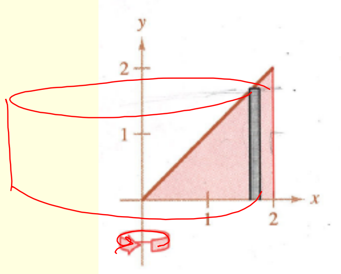
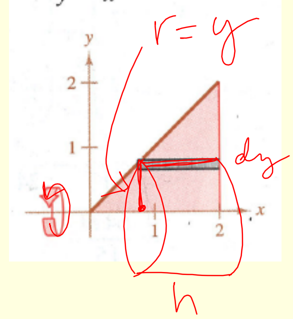
In Exercises 1–20, use the shell method to find the volume of the solid generated by revolving the given plane region about the indicated line.

$y = 4x^2 + 3$

1. $y = x$

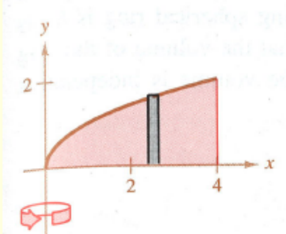


3. $y = x$



$$2\pi \int_0^2 y(2-y)\, dy$$

5. $y = \sqrt{x}$

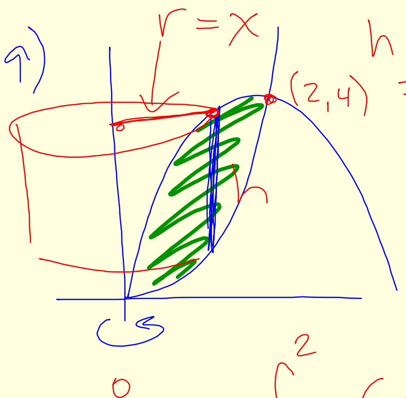


7. $y = x^2$, $y = 0$, $x = 2$, about the y-axis

9. $y = x^2$, $y = 4x - x^2$, about the y-axis

11. $y = x^2$, $y = 4x - x^2$, about the line $x = 4$

9)



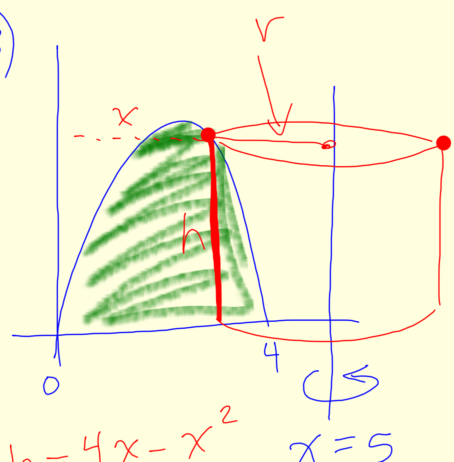
$h = 4x - x^2 - x^2$
$= 4x - 2x^2$

$$2\pi \int_0^2 x(4x - 2x^2)\,dx$$

13. $y = 4x - x^2$, $y = 0$, about the line $x = 5$

15. $y = 4x - x^2$, $x = 0$, $y = 4$, about the y-axis

13)



$$2\pi \int_0^4 (5-x)(4x-x^2)\,dx$$

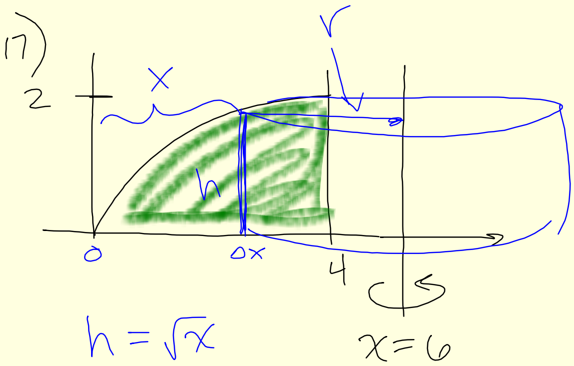
$h = 4x - x^2$

$r = 5 - x$

17. $y = \sqrt{x}$, $y = 0$, $x = 4$, about the line $x = 6$

19. $x = y - y^2$, $x = 0$, about the line $y = 2$

17)



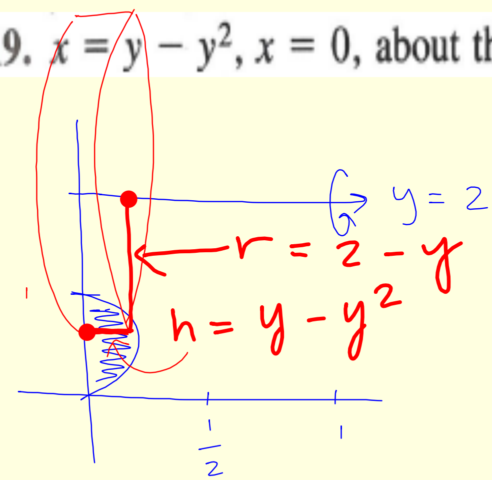
$$2\pi\int_0^4 (6-x)\,x^{1/2}\,dx$$

$h = \sqrt{x}$

$r = 6 - x$

17. $y = \sqrt{x}$, $y = 0$, $x = 4$, about the line $x = 6$

19. $x = y - y^2$, $x = 0$, about the line $y = 2$



$$2\pi\int_0^1 (2-y)(y-y^2)\,dy$$

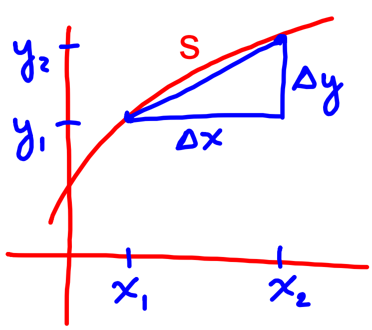
6.4 Arc length

-Surface of revolution

-Area of surface of revolution

Let s = Arc Length

We can estimate the length of a curve by considering line segments and the distance formula, making those line segments smaller and smaller, then summing them up!

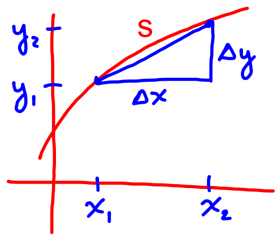


$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Let s = Arc Length

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$



Approximating the length of the curve by summing up very small line segments:

$$\sum \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}\ \Delta x}{\sqrt{(\Delta x)^2}}$$

$$\sum \sqrt{\frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}}\ \Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Taking the limit as $\|\Delta\| \to 0$, $(n \to \infty)$, we obtain

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,(\Delta x_i)$$

Since $f'(x)$ exists for each x in (x_{i-1}, x_i), the Mean Value Theorem guarantees the existence of c_i in (x_{i-1}, x_i) such that

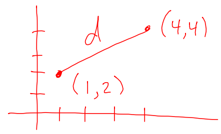
$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) \quad \Rightarrow \quad \boxed{\frac{\Delta y_i}{\Delta x_i} = f'(c_i).}$$

Moreover, since f' is continuous on $[a, b]$, we know that $\sqrt{1 + [f'(x)]^2}$ is also continuous (and hence integrable) on $[a, b]$ and we have

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(c_i)]^2}\,(\Delta x_i) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We call s the **arc length** of f between a and b.

Test it:



$$d=\sqrt{(4-1)^2+(4-2)^2}$$
$$=\sqrt{9+4}$$
$$=\sqrt{13}$$

$$S=\int_a^b \sqrt{1+[f'(x)]^2}\,dx \rightarrow f'(x)=m=\frac{2}{3}$$

$$=\int_1^4 \sqrt{1+\left(\tfrac{2}{3}\right)^2}\,dx$$

$$\int_1^4 \sqrt{\frac{13}{9}}\,dx$$

$$\frac{1}{3}\int_1^4 \sqrt{13}\,dx$$

$$\frac{1}{3}\Big[\sqrt{13}\,x\Big]_1^4$$

$$\frac{1}{3}\left(4\sqrt{13}-\sqrt{13}\right)$$

$$\frac{1}{3}\left[3\sqrt{13}\right]=\sqrt{13}\ \checkmark$$

You try: Find the exact length of the curve

$$y=\frac{4\sqrt{2}}{3}x^{3/2}-1 \quad \text{on } [0,1]$$

$$y'=2\sqrt{2x} \qquad S=\int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

$$S=\int_0^1 \left(1+\left(2\sqrt{2x}\right)^2\right)^{1/2}dx$$

$$\frac{1}{8}\int_0^1 8(1+8x)^{1/2}dx \qquad u=1+8x,\ du=8\,dx$$

$$=\frac{1}{8}\left[\frac{2u^{3/2}}{3}\right]_1^9 \qquad \text{for } x=0\rightarrow u=1,\ x=1\rightarrow u=9$$

$$=\frac{1}{8}\cdot\frac{2}{3}\Big[u^{3/2}\Big]_1^9$$

$$=\frac{1}{12}(27-1)$$

$$=\boxed{\frac{13}{6}}$$

Another: Find the length of the curve

$f(x) = \frac{x^3}{6} + \frac{1}{2x}$ on $\left[\frac{1}{2}, 2\right]$

$f'(x) = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$

$2x^2\left(-\frac{1}{x^2}\right)$

$\int_{1/2}^{2} \sqrt{1 + \left(\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right)^2}$

$\int \sqrt{\frac{4}{4} + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\, dx$

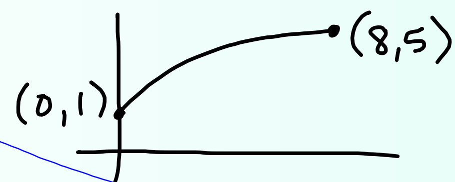
$\frac{1}{2}\int_{1/2}^{2} \sqrt{\left(4 + x^4 - 2 + \frac{1}{x^4}\right)}\, dx$

$\frac{1}{2}\int_{1/2}^{2} \sqrt{\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx$

$\frac{1}{2}\int_{1/2}^{2} \sqrt{\left(x^2 + \frac{1}{x^2}\right)^2}\, dx$

You can also find the length of the curve with respect to y:

$x = (y-1)^{3/2}$



$\int_{1}^{5} \sqrt{1 + (\qquad)^2}\, dy$ $\qquad \frac{3}{2}(y-1)^{1/2}(1)$

Definition of a <u>Smooth Curve</u>: The graph of a continuously differentiable function.

DEFINITION OF ARC LENGTH

If the function given by $y = f(x)$ represents a smooth curve on the interval $[a, b]$, then the **arc length** of f between a and b is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Similarly, for a smooth curve given by $x = g(y)$, the **arc length** of g between c and d is given by

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

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1 - 13 odd